

INVESTIGATION OF POWER-LAW DAMPING/DISSIPATIVE FORCES

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Abstract. The properties of a one space-dimension, one particle dynamical system under the influence of a purely dissipative force are investigated. Assuming this force depends only on the velocity, it is demonstrated, in contrast to the case of linear damping, that there exist dissipative forces for which the particle “stops” in a finite time. It is also shown, by an explicit example, that other dissipative forces exist such that they produce dynamics in which the particle achieves zero velocity only after an infinite distance has been traveled. Possible applications of these results to more complex situations are discussed.

1. INTRODUCTION

Dissipation and/or damping forces play very significant roles in the modeling and analysis of many physical systems. [1, 2, 3, 4, 5, 6] Dissipation is generally the term used for situations where a conservative system is acted upon by friction and/or viscous forces, while damping is often used in cases where the relevant system undergoes oscillatory motions. However, in many instances, no such clear distinctions can be made. The major characterizations of such forces separate them into three broad classes [2, 3, 6]: viscous damping, aero-dynamic drag, and dry friction, although, other possible types of forces may be defined when such concepts are of value, an example being hysteretic damping. [2] Within the context of the issues discussed in this presentation, we will simply use the term “damping force.”

If we consider a localized conservative physical system interacting with an energy absorbing environment, then the system will evolve in time to a locally stable stationary state. [7] An

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example of such a system is an unforced pendulum, set in motion, in air. Eventually, it reaches a state of equilibrium where it hangs vertically, with no angular velocity. [2] However, for this and many other systems, it is standard practice, in first approximation, to represent the damping force by a term linear in the velocity. An important, nontrivial, and also nonlinear case is the unforced Duffing equation. [2, 3, 4, 6]

$$m\ddot{x} + 2\varepsilon\dot{x} + \omega^2x + kx^3 = 0,$$

where $(m, \varepsilon, \omega^2, k)$ are constant parameters, all of which are taken to be positive, except for ω^2 and k , which may be of either sign. [6] (The "dot" is used to indicate the time-derivative, i.e., $\dot{x} = \frac{dx(t)}{dt}$.) The general qualitative features of solutions to this equation are well known. [2, 4, 6] For $x(0) \neq 0$ and either $\dot{x}(0) = 0$ or $\dot{x}(0) \neq 0$, the Duffing oscillator undergoes damped oscillations, with the rest or equilibrium state being achieved only in the limit $t \rightarrow \infty$. However, observations of actual nonlinear oscillatory systems lead to the conclusion that the oscillations generally take place only over a finite time interval, T . For $t > T$, the system is at rest in its equilibrium state,

$$x(t) = 0, \quad \dot{x}(t) = 0, \quad t > T,$$

and the oscillations have stopped. Other common mathematical forms for the damping force, which we assume is only a function of the velocity, \dot{x} , and consists of sums of terms proportional to integer power of \dot{x} , give the same result [2, 3, 4, 6] as for linear damping.

The main goal of this presentation is to determine if damping forces exist such that, at least in certain model systems, the motion or dynamics ends after a finite time interval. We show that for a certain class of damping forces, an affirmative answer can be given. To do this, we study the dynamics of a one-dimensional system: a particle, acted on by a single, purely damping force, where this force is taken to be a power of the velocity, and the value of the power is a non-negative real number. While the focus is on a particular one-dimensional system, the derived results and related conclusions are of general validity for more complex systems.

In the next section, we briefly state the various assumptions that must be placed on a valid damping force and explain how they arise from knowledge and understanding of actual physical systems. Section III gives the model differential equation and its solutions for various values of

a parameter α , which characterizes the damping force. The results obtained in section III and their implications are discussed in section IV. Finally, in section V, our major conclusions are briefly restated and possible applications are presented.

It should be noted that the results of this article has direct applicability and relevance to various topics arising in introductory and advance physics courses, and in many practical applications. Generally, the effects of damping are studied by assuming the corresponding force is a linear function of the velocity. While there are circumstances where such a form may be justified, [1, 2, 3, 4, 5, 6] the main reason for using this particular functional form is that it often leads to linear differential equations which can then be easily solved by techniques [7] well known to the students and their teachers, and researchers.

2. PROPERTIES OF DAMPING FORCES

Consider a particle of mass, m , moving in the direction of the positive x -axis, such that at time $t = 0$, it is located at $x = 0$, with velocity $v_0 > 0$. Assume that the only force acting on it is a (viscous) damping force which depends only on the velocity. The equation of motion, [7] for $m = 1$, is

$$\ddot{x} = -f(\dot{x}), \quad x(0) = 0, \quad \dot{x}(0) = v_0 > 0, \quad (1)$$

where $f(v)$, $v = \dot{x}$, is the negative of the damping force. Physical restrictions place the following constraints on $f(v)$:

- i) $f(v)$ is continuous in v ;
- ii) $f(-v) = -f(v)$;
- iii) $f(v) > 0$, if $v > 0$.

Condition i) is a reasonable mathematical requirement and combined with ii) gives the result $f(0) = 0$. Condition ii) is consistent with the fact that damping forces change signs if the direction of motion is changed to its opposite. Finally, iii) is what is expected of any purely damping force, i.e., it should always act in the direction opposite to the motion for any velocity.

Another characterization of $f(v)$ is to consider its impact on the energy of the particle. Without the damping force, the particle (with $m = 1$) has a total energy equal to its kinetic energy

$KE = v^2/2$. If Eq. (1) is rewritten to the system form

$$\dot{x} = v, \quad \dot{v} = -f(v),$$

then

$$\frac{d(KE)}{dt} = -vf(v) \leq 0.$$

Therefore, it follows that the KE decreases to zero, as $t \rightarrow \infty$, and, as a consequence, the effect of $f(v)$ is to have the energy continuously decrease to zero.

The class of damping functions investigated here are defined by a parameter α , and have a power-law dependence on v . The form selected is

$$f(v) = [\lambda \operatorname{sgn}(v)]|v|^\alpha, \quad \lambda > 0, \quad \alpha > 0, \quad (2)$$

where λ is a positive constant, α is real and positive, and $\operatorname{sgn}(v)$ is the “sign-function” [8],

$$\operatorname{sgn}(v) = \begin{cases} +1, & v > 0, \\ 0, & v = 0, \\ -1, & v < 0. \end{cases}$$

Inspection of Eq. (2) shows that it satisfies the three conditions i), ii), and iii). However, for our “toy” physical system, the $\operatorname{sgn}(v)$ does not need to be indicated since the velocity is positive and never changes direction, i.e., combining Eqs. (1) and (2), we obtain

$$\ddot{x} = -\lambda(\dot{x})^\alpha, \quad x(0) = 0, \quad \dot{x}(0) = v_0 > 0. \quad (3)$$

3. CALCULATIONS

The dynamics of Eq. (3) will now be determined. To do so, five intervals of α -values are examined. They are A) $\alpha > 2$, B) $\alpha = 2$, C) $1 < \alpha < 2$, D) $\alpha = 1$, and E) $0 < \alpha < 1$.

3.1. $\alpha > 2$. Let $\alpha = 2 + \beta$, where $\beta > 0$. Therefore, Eq. (3) becomes

$$\dot{v} = -\lambda v^{2+\beta}, \quad v(0) = v_0 > 0. \quad (4)$$

An elementary integration gives the solution [7, 8]

$$v(t) = \frac{v_0}{\left[1 + \lambda(1 + \beta)v_0^{(1+\beta)}t\right]^{\frac{1}{1+\beta}}}. \quad (5)$$

Note that for large t , $v(t)$ is asymptotic to

$$v(t) \sim \frac{c_1}{t^{\left(\frac{1}{1+\beta}\right)}}, \quad (6)$$

where c_1 is expressible in terms of λ , β , and v_0 . This result implies that

$$\lim_{t \rightarrow \infty} v(t) = 0. \quad (7)$$

The displacement function, $x(t)$, is given by

$$x(t) = \int_0^t v(z)dz, \quad (8)$$

which automatically incorporates the condition $x(0) = 0$. However, while the substitution of Eq. (5) into this integral will allow the explicit calculation of $x(t)$, our interest is in $x(\infty)$, i.e.,

$$x(\infty) = \lim_{t \rightarrow \infty} x(t) \equiv \int_0^\infty v(z)dz. \quad (9)$$

From the calculus [9], it follows that the result given in Eq. (6) implies that the singular integral in Eq. (9) is unbounded. Thus, while $v(t) \rightarrow 0$, as $t \rightarrow \infty$, the particle's motion carries it an infinite distance from the origin.

3.2. $\alpha = 2$. For this case, the equation of motion is

$$\dot{v} = -\lambda v^2, \quad v(0) = v_0 > 0, \quad (10)$$

and its solution is

$$v(t) = \frac{v_0}{1 + \lambda v_0 t}. \quad (11)$$

Likewise, $x(t)$ is

$$x(t) = \int_0^t v(z)dz = \frac{\ln(1 + \lambda v_0 t)}{\lambda}, \quad (12)$$

and $x(\infty) = \infty$.

3.3. $1 < \alpha < 2$. Let $\alpha = 1 + \gamma$, where $0 < \gamma < 1$, and the equation of motion is

$$\dot{v} = -\lambda v^{1+\gamma}, \quad v(0) = v_0 > 0; \quad (13)$$

an elementary integration gives

$$v(t) = \frac{v_0}{(1 + \lambda \gamma v_0^\gamma t)^{1/\gamma}}. \quad (14)$$

Therefore,

$$x(t) = v_0 \int_0^t \frac{dz}{(1 + \lambda \gamma v_0^\gamma z)^{1/\gamma}} = \left[\frac{v_0^{(1-\gamma)}}{\lambda(1-\gamma)} \right] \left\{ 1 - \frac{1}{(1 + \lambda \gamma v_0^\gamma t)^{1-\gamma/\gamma}} \right\}, \quad (15)$$

and

$$x(\infty) = \frac{v_0^{(1-\gamma)}}{\lambda(1-\gamma)} < \infty. \quad (16)$$

Thus, it follows that the velocity decreases to zero and the distance traveled by the particle is finite.

3.4. $\alpha = 1$. For this case,

$$\dot{v} = -\lambda v, \quad v(0) = v_0 > 0, \quad (17)$$

and the solutions for $v(t)$ and $x(t)$ are, respectively,

$$v(t) = v_0 e^{-\lambda t}, \quad (18)$$

$$x(t) = \left(\frac{v_0}{\lambda} \right) (1 - e^{-\lambda t}). \quad (19)$$

The velocity and displacement, respectively, monotonically decrease and increase to the values $v(\infty) = 0$ and $x(\infty) = v_0/\lambda$.

3.5. $0 < \alpha < 1$. Let $\alpha = 1 - \delta$, where $0 < \delta < 1$. Then the equation of motion is

$$\dot{v} = -\lambda v^{1-\delta}, \quad v(0) = v_0 > 0 \quad (20)$$

and its formal solution is

$$v(t) = [v_0^\delta - \delta \lambda t]^{1/\delta}.$$

Note that this $v(t)$ can become complex valued for $t > t^* = v_0^\delta / \delta \lambda$ and also observe that since Eq. (20) is a nonlinear differential equation, $v(t) = 0$, is also a valid, nontrivial solution. [7]

The implication of these facts is that $v(t)$ is expressed as the following piecewise, continuous function

$$v(t) = \begin{cases} [v_0^\delta - \lambda \delta t]^{1/\delta}, & 0 < t \leq t^* = \frac{v_0^\delta}{\delta \lambda}, \\ 0, & t > t^*. \end{cases} \quad (21)$$

Likewise, $x(t)$ is

$$x(t) = \left[\frac{1}{\lambda(1+\delta)} \right] \left\{ v_0^{1+\delta} - [v_0^\delta - \lambda \delta t]^{\frac{1+\delta}{\delta}} \right\}, \quad (22a)$$

$$0 < t \leq t^*$$

and

$$x(t) = \frac{v_0^{1+\delta}}{\lambda(1+\delta)}, \quad t > t^*. \quad (22b)$$

Consequently, the particle stops moving in a finite time, after having traveled the finite distance given in Eq. (22b)

4. RESULTS AND DISCUSSION

We now summarize the results obtained in section III. Several of the outcomes are surprising and have not been either explicitly stated and/or discussed in any of the literature known to the authors.

To begin, observe that both of the following differential equations are mathematically equivalent:

$$\ddot{x} = -[\lambda \operatorname{sgn}(\dot{x})](\dot{x})^\alpha, \quad x(0) = 0, \quad \dot{x}(0) = v_0 > 0, \quad (23a)$$

$$\dot{v} = -[\lambda \operatorname{sgn} v](v)^\alpha, \quad v(0) = \dot{x}(0) = v_0 > 0, \quad (23b)$$

since $v = \dot{x}$. In the discussion below, both forms will be used. Also, it should be noted that the cases where $\alpha = 1$ or 2 are used quite extensively in the modeling of physical systems where damping plays an important role. The $\alpha = 1$ value is applied to situations where so-called “linear viscous damping” occurs and is used often in the modeling of nonlinear vibrations and oscillations. [1, 2, 3, 4, 6] However, $\alpha = 2$, i.e., quadratic damping, generally appears in fluid

dynamic problems at high Reynold's number; [2, 3, 5, 6] also see Ref. [6], (section 6.2.2), for more details and additional references to the appropriate literature.

The $\alpha = 0$ case provides a model of dry friction; in general, this type of friction occurs when two solid surfaces are in contact and moving relative to each other. [2, 3, 4] This case is not considered in our studies since the full, anti-symmetric damping force corresponding to this situation is

$$f(v) = \lambda \operatorname{sgn}(v), \quad \alpha = 0,$$

and consequently is not continuous at $v = 0$. However, by considering a function such as

$$\dot{v} = -\lambda \tanh(av),$$

where a is a positive parameter, and then taking the limit as $a \rightarrow \infty$, the $\alpha = 0$ case is reproduced.

We now summarize the results of the calculations presented in section III; for all cases, the initial conditions are those indicated in Eqs. (23):

a) For $\alpha \geq 2$, the velocity, $v(t)$, monotonically decreases from $v(0) = v_0 > 0$ to zero, while $x(t)$ monotonically increases to $x(\infty) = \infty$, where $x(\infty) = \lim_{t \rightarrow \infty} x(t)$.

b) If $1 \leq \alpha < 2$, then $v(t)$ monotonically decreases from $v(0) = v_0 > 0$ to zero. However, $x(t)$ achieves a finite maximum value, i.e.,

$$\lim_{t \rightarrow \infty} x(t) = x_{\max} < \infty,$$

and it takes an infinite length of time to reach this limiting value.

c) The α values in the interval, $0 < \alpha < 1$, all give rise to motions for which $v(t)$ decreases and $x(t)$ increases; however, both variables reach their limiting values in a finite time, i.e., there exists a time, t^* , such that $v(t^*) = 0$ and $v(t)$, for $t > t^*$, is zero; and $x(t)$ increases to a maximum value x_{\max} at $t = t^*$ and for $t > t^*$, $x(t) = x_{\max}$.

The third case is the interesting and important one. Since actual conservative physical systems, acted upon by only external damping forces, are observed/found to reach a stable equilibrium state in a finite time, then if our results are of general applicability, it follows that for the class of damping forces considered here, only the α -parameter interval, $0 < \alpha < 1$, should be used to model the required dissipative or damping force for such systems. Further, because

most dynamic models are represented as second-order differential equations, this force should be expressed in a functional form such as that given on the right-side of Eq. (23a).

An equation of particular importance for the modeling of a broad range of nonlinear oscillatory phenomena is the Duffing equation. [6] It provides a useful and realistic model for pendulum-type problems, and for a variety of issues involving nonlinear vibrations/oscillations of beams, cables, electrical circuits, and plates; see Kovacic and Brennan, Chapter 2. Another recent application of the Duffing equation is to the analysis and understanding of the mechanical and electro-mechanical properties of graphene. [10, 11] Single and multi-sheets of this material are being studied for a wide range of possible measuring devices which may be used in new technological applications.

In standard form the Duffing equation can be written as

$$m\ddot{x} + \varepsilon f(\dot{x}) + \omega^2 x + \beta_1 x^2 + \beta_2 x^3 = 0 \quad (24)$$

for the case where there is no external driving force; [2, 3, 6] $(m, \varepsilon, \omega^2)$ are non-negative, while (β_1, β_2) may be of either sign or zero. However, there are also important systems where the sign of the $\omega^2 x$ term is taken to be negative. [6] To date, the major emphasis has been on the study of Eq. (24) where $f(v)$ is that given by Eq. (2), with $\alpha = 1$ or 2; see Ref. [6].

For forced oscillations, a large research literature exists on the case where a forcing term $F \cos(\Omega t)$ is placed on the right-side of Eq. (24), i.e.,

$$m\ddot{x} + \varepsilon f(\dot{x}) + \omega^2 x + \beta_1 x^2 + \beta_2 x^3 = F \cos(\Omega t), \quad (25)$$

where Ω is the external driving (angular) frequency. Most of these studies are based on using $\alpha = 1$ for the damping force. [2, 3, 4, 6] For other positive, integer values, a technique called equivalent viscous damping may be used to effectively linearize the nonlinear damping force (see Ref.[6]; section 6.2.3 and Ref.[12]). In general, the major outcome of these calculations is that different values of α produce differences in the properties of the forced oscillatory states (see Ref. [6], section 6.8). However, the overall qualitative features for α integer, but greater than one, is similar to the case where $\alpha = 1$.

Based on the results of our study, it might be useful to consider values of α belonging to the interval $(0, 1)$ for the modeling of physical systems having damping. Note that if α is selected

to be

$$\alpha = \begin{cases} \frac{2n+1}{2m+1}, \\ (n, m) \text{ non-negative integer,} \\ n < m, \end{cases} \quad (26)$$

then the damping function

$$f(v) = \lambda v^{\frac{2n+1}{2m+1}}, \quad (27)$$

is automatically an odd function [13] and the $\text{sgn}(v)$ function, see Eq. (2), is not required in the specification of $f(v)$. Elastic force functions corresponding to the case, $n = 0$ and $m = 1$, has been extensively investigated for the case of undamped oscillators [13]. The simplest type of such an oscillator is [13]

$$\ddot{x} + x^{1/3} = 0. \quad (28)$$

An extension of this type of nonlinear term to the damping force produces the following version of the Duffing equation

$$m\ddot{x} + \varepsilon \dot{x}^{1/3} + \omega^2 x + \beta_1 x^2 + \beta_2 x^3 = F \cos(\Omega t). \quad (29)$$

It would be of considerable theoretical and practical value to investigate the unforced version of Eq. (29), with $\beta_1 \geq 0$ and $\beta_2 > 0$, and determine whether the damped oscillations decay to zero in a finite time. If so, this would provide justification to not use damping forces depending solely on integer powers of α . Further, this result would then require a detailed analysis of the physical mechanisms relating to various damping forces which might give values of α such that $0 < \alpha < 1$. In particular, a very important result would be to determine whether α must take on the fractional values indicated in Eq. (26), or just be a real number.

It should be noted that previous work by Mickens et al. [14] on a cube-root damped harmonic oscillator

$$\ddot{x} + x = -2\varepsilon \dot{x}^{1/3}, \quad (30)$$

showed that the oscillations decreased to zero in a finite time. This implies that only a finite number of oscillations take place before the equilibrium state, $x = 0$, is reached. Exactly, the same conclusion is reached for the singular case, i.e., $\alpha = 0$, which represents dry friction or

Coulomb damping, [2, 3, 4] and for the other cases where the $\frac{1}{3}$ in Eq. (30), is replaced by α , such that $0 < \alpha < 1$.

5. CONCLUSIONS

Within the context of a model system, in which the only force acting on a particle is a damping force, it has been shown that a class of dissipative forces exist such that the dynamic behavior ends in a finite time. An important aspect of this result is that it agrees with what is observed for the actual behavior of physical systems, i.e., the oscillations or motions stop in a finite time. We believe the use of such damping forces in the construction of mathematical models of physical systems may provide more realistic models than are currently given by using a linear or quadratic (in velocity) damping term.

Finally, it should be indicated that in the modeling of a single particle system, in one space dimensions, a very general damping and/or dissipative force takes the form

$$f(x, v) = [\text{sgn}(v)] \left[a_1 |v|^\alpha + a_2 |v|^2 \right] + [b_1 + b_2 x^2 + b_3 v^2] v, \quad (31)$$

where the parameters $(a_1, a_2, b_1, b_2, b_3)$ are all non-negative, and $0 < \alpha < 1$. This functional form includes all of the standard proposed power-law laws, including the fractional power force.[1, 3, 5, 6] By applying the method of "dominant balance" [15], it follows that for small v

$$f(x, v) \simeq a_1 [\text{sgn}(v)] |v|^\alpha, \quad (32)$$

and, consequently, finite time dynamics is expected to result.

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